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Geometry and Quantization

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Abstract

Gravity is treated geometrically in terms of nonlinear realizations of $GL(4, \mathbb{R})$ with particular reference to almost complex structures. This approach is used to carry out a Bargmann–Segal type quantization of space-time via the vector and spinor structures of the tangent space that formulates the theory of measurement as a quantum theory quantized in terms of a basic unit of length that appears in a new uncertainty relation. The theory is also used to discuss the gauge conditions for quantum gravity and the Kostant theory of quantization applied using a line bundle with structure group $GL(2, \mathbb{C})/SL(2, \mathbb{C})$.

1. *Introduction*

This paper is concerned with exploration of the relation between certain abstract aspects of quantization and geometry, and with the application of this geometry to a formal quantization of space-time and of a part, the trace, of the gravitational field.

There are several motivating factors for this study:

(1) Geometry enters into free field theory through the commutation relations.

(2) If gravity is essentially geometric in nature, then the connection of complex and symplectic structures with quantization might be relevant to quantum gravity.

(3) These geometric structures constitute nonlinear realizations of $GL(4, \mathbb{R})$, and one such realization has already been shown by Isham *et al.* (Isham *et al.*, 1971a) to be helpful for the regularization of divergences of quantum electrodynamics.

(4) Complex structures are closely related to spinor structures and hence to the Newman–Penrose formalism (Penrose, 1960; Newman and Penrose, 1962), on which the most successful classical treatments of gravitational radiation have been based.

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(5) In writing manifestly $SL(2,\mathbb{C})$ covariant wave functions for fields of definite mass and spin $\geqslant 1$ the actual representation of $SL(2,\mathbb{C})$ is arbitrary, but in the presence of interactions the different representations are inequivalent.

(6) Gravity is a self-interacting field, and since the motivation for its quantization comes from the operator character of the right-hand side of the Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}$$

when other fields are quantized, then the part of the gravitational field belongs to the (0, 0) and (1, 1) representations of $SL(2,\mathbb{C})$ whilst the curvature of space-time renders the usual Fourier decomposition on $\mathbb{R}^4$ inadmissible.

(7) Changes in topology resulting from quantum fluctuations of the curvature require some form of quantization of space-time itself on the scale of the Planck length.

(8) The formalism presented here is relevant to a more general theory by the author in which the base space of a fiber bundle enters at a prerelativistic level.

As a preliminary to investigation of quantum geometry a brief review of gauge invariance approaches to gravity is presented in Section 2 in order to justify the use of the formalism adopted here. The most important point made is the distinction between two different $GL(4, \mathbb{R})$ groups, which is vital to the subsequent development of the theory.

With regard to (2), the rigorous geometrization of Bohr–Sommerfeld type quantization has been carried out by Kostant (Kostant, 1970b) in terms of the curvature of line bundles and the integrality of cohomology classes, whilst the idea of field quantization in terms of complex and symplectic structures has been considered by Segal in a number of publications (Segal, 1960, 1962, 1963, 1969) and applied to a rigorous quantization of a free field. Section 3 begins with an outline of the basic geometry of these schemes and considers its application to the pseudo-Riemannian geometry of space-time. This leads to a model of a graviton as a nonlinear spinor closely related to the linear Penrose spinor, and some comparison is made of this with the gauge conditions involved in various approaches to the quantization of gravity, which involves some discussion of momentum space orbits for $\mathbb{R}^4 \circledS GL(4, \mathbb{R})$ and its subgroups.

The treatment of quantization as a form of analyticity restricts consideration of representations of $SL(2,\mathbb{C})$ to those by polynomials which are realized geometrically in terms of spinor and tensor bundles. Section 4 contains the use of the Bargmann–Segal technique for the quantization of tangent and spinor spaces at each point of space-time. Of these, the former essentially exhibits the theory of measurement of length as a (formally) quantized process in which a basic unit of length is introduced in a Lorentz covariant fashion; this is accompanied by a new uncertainty principle in which fluctuations in measurements of lengths in directions mutually conjugate with respect to the complex structure determined by the light-cone are of the order of the basic unit. The quantization of spinors gives points and antipoints as the basic quanta of geometry with properties resembling spin-1/2 particles.

A rigorous approach to the group-theoretic gauge conditions required for field quantization would involve the application of the Kostant approach to space-time regarded as a symplectic manifold, and in Section 5 this is outlined. The line bundle involved has standard fiber $GL(2,\mathbb{C})/SL(2,\mathbb{C})$ so that the integrality condition for quantization is satisfied, but owing to the lack in general of globally Hamiltonian vector fields the main effect of this is to rewrite the familiar problems of quantization in the absence of Killing vector fields in a different language.

2. *Gauge Theories*

Before reviewing gauge field descriptions of gravity it must be emphasized that following Weinberg's (Weinberg, 1964b) demonstration that gauge invariance and the principle of equivalence are necessary consequences of the Lorentz invariance of S-matrix elements for soft helicity two zero mass scattering rather than necessarily the source of such particles, it may be that geometric concepts have no fundamental significance in relation to gravity.

Historically gauge fields were first introduced by Weyl for dilatation invariance and Lorentz invariance of the vierbein (Weyl, 1918, 1929, 1950) in his attempts to extend general relativity, but the first description of gravity itself as a gauge field was by Utiyama (Utiyama, 1956). Utiyama derived the field equations of general relativity by the Yang-Mills technique for invariance of the Lagrangian under homogeneous Lorentz transformations of the second kind; however, as pointed out by Kibble (Kibble, 1961), the metric tensor was introduced through the inconsistent use of curvilinear coordinates, whilst the assumption of a symmetric connection was made arbitrarily. Kibble overcame these defects with an analogous theory of Poincaré gauge invariance leading to the field equations of general relativity together with additional equations coupling torsion to spin as in Weyl's theory (Weyl, 1929, 1950) and the Einstein-Cartan theory (Cartan, 1923). The use of $\mathbb{R}^4$ gauge transformations has been subject to some criticisms by Trautman (Trautman, 1970), mentioned below, and the same author (Trautman, 1972) has also derived the Einstein-Cartan field equations by variation of the Lagrangian with respect to affine frames.

Closely related to the Lorentz invariance approach is the use of $SL(2,\mathbb{C})$ invariance by Carmeli and Isham (Carmeli, 1972; Isham *et al.*, 1972). Carmeli showed how to obtain the Newman-Penrose equations from a Yang-Mills-like theory using a quadratic Lagrangian, but with the usual noncovariant Yang-Mills field replaced by a covariant field B_μ formed from the spin coefficients

$$B_\mu = \sigma_\mu^{ab'} B_{ab'}$$

where σ_μ are van der Waerden symbols and B spin coefficients. In this the dynamical role of the curvilinear coordinates appears slightly obscure, but it is manifest in the theory of Isham *et al.* in which they enter through the contracted vierbein $L^\mu \equiv L_a^\mu \gamma^a$ which is canonically conjugate to B_μ with respect to various Lagrangians of which the simplest is $i\,\mathrm{Tr}[L^\mu, L^\nu]\,B_{\mu\nu}$, where

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu + i\,[B_\mu, B_\nu]$$

The extra spin force of the Weyl theory associated with torsion is not discussed, the objective of the work being to generalize $SL(2,\mathbb{C})$ gauge techniques to $SL(6,\mathbb{C})$ and f-g mixing.

A quite different approach due to Isham *et al.* (Isham *et al.*, 1971b) is to try to describe gravitation as due to Goldstone bosons for $GL(4,\mathbb{R})/O(1,3)$, and this had the major result of regularizing divergences of quantum electrodynamics, whilst providing the Einstein field equations.

The relationship of these theories to the present work can be seen by attempting to express them in terms of fiber bundles. The fact that gauge theories can be expressed in terms of principal fiber bundles whose structure groups are the gauge groups is well known (Lubkin, 1963; Loos, 1965), but it is necessary in geometry to distinguish between frame bundles and arbitrary principal fiber bundles with isomorphous structure group (Trautman, 1970), since the bundle space of the former, but not the latter, is rigidly fixed to the base space by the canonical (or soldering) form. Geometric theories of gravity and possible generalizations involving first-order G structures are defined in terms of the bundle of affine frames with structure group $\mathbb{R}^4$ Ⓢ $GL(4,\mathbb{R})$ and its subbundles based on a space M which is locally $\mathbb{R}^4$. The canonical form $\theta(X)$ for any first-order frame bundle P over M with projection π is defined by

$$\theta(X) = u^{-1}[\pi(X)], \qquad X \in T_u(P), \quad u \in P$$

where $T_u(P)$ is the tangent space to the bundle at u and u is considered as a linear mapping of $\mathbb{R}^4$ onto $T_{\pi(u)}(M)$. $\theta(X)$ is a tensorial form of type $(GL(4,\mathbb{R}), \mathbb{R}^4)$ and corresponds to a tensor of type $(1,1)$, which represents the field of identity transformations of the tangent spaces $T_x(M)$, $x \in M$.

A connection form $\tilde{\omega}$ in the bundle $P(M)$ of affine frames with structure group $\mathbb{R}^4$ Ⓢ G, where G is either $GL(4,\mathbb{R})$ or a closed Lie subgroup, can be decomposed into the semidirect Lie algebra sum of the connection form ω for the corresponding linear bundle and an $\mathbb{R}^4$-valued 1-form ϕ:

$$\tilde{\omega} = \omega + \phi$$

Similarly the curvature form $\tilde{\Omega}$ splits as follows:

$$\tilde{\Omega} = \Omega + D\phi$$

where D denotes the horizontal exterior derivative and $\Theta = D\theta$ is the torsion form.

The theories of Utiyama and Kibble are both related to the bundle of orthonormal affine frames with structure group $\mathbb{R}^4$ Ⓢ $SO(1,3)$ with the components of the Yang-Mills fields $A_{\mu j}^{\ i}$ and $h_\mu^{\ k}$ corresponding to the coefficients of the two parts ω and ϕ of $\tilde{\omega}$, with Latin indices referring to the fiber space and Greek to the base, which these authors took to be flat space-time. Utiyama's theory is not internally consistent, whilst in Kibble's theory

Trautman's criticism is essentially that the canonical form θ provides a preferred value for ϕ and hence (up to a change of parametrization) a set of values $\delta_\mu{}^k$ for $h_\mu{}^k$. In this strictly geometric interpretation $h_\mu{}^k$ defines an arbitrary set of coordinates but not a dynamical field.

As regards $GL(4, \mathbb{R})$, it has been pointed out by Isham that the field equations of general relativity are invariant under two distinct $GL(4, \mathbb{R})$ groups, one generated by nonsingular infinitesimal Jacobian matrices of general coordinate transformations and related to the 1-form ϕ as above, and the second the group of linear transformations of the basis frames. Formally, breaking of either of these symmetries to $O(1, 3)$ appears to lead to Goldstone bosons of the vierbein gravitational field, but in fact geometrically they are quite distinct; there are two major differences, the first being that the general coordinate transformations do not contain transformations of the spin frames [or indeed other interesting groups of frame transformations such as $GL(2, \mathbb{C})$ or $Sp(2, \mathbb{R})$] whilst the frame transformations do, and the second is that the canonical form breaks the coordinate transformations down to the identity element without influencing the frame transformations. Spinors transform linearly under $GL(4, \mathbb{R})$ coordinate transformations and nonlinearly under the frame group, but since the existence of the canonical form means that a change of basis frame induces a preferred coordinate transformation it is necessary for linear and nonlinear spinor theories to be consistent, and this was shown by Isham *et al.* (Isham *et al.*, 1971b).

According to Kobayashi and Nomizu (Kobayashi and Nomizu, 1963) a Riemannian structure is uniquely determined by a reduction of the bundle of general linear frames with group $GL(4, \mathbb{R})$ to that of orthonormal frames with group $O(4)$, whilst pseudo-Riemannian structures are obtained apart from uniqueness problems due to noncompactness (Isham *et al.*, 1971b) by reduction to $O(1, 3)$. The gauge theory most directly related to general relativity is thus that of $GL(4, \mathbb{R})/O(1, 3)$, and it is well known that the field equations can be obtained from contraction of the tensorial version of the Bianchi identity $D\Omega = 0$. The effect of introducing $SL(2, \mathbb{C})$ or $O(1, 3)$ gauge transformations of the second kind is to relax the requirement that the connection be the unique Riemannian connection, and it produces Weyl's extended theory, which contains torsion (Sciama, 1964). Such gauge transformations on flat space were considered by Rodichev (Rodichev, 1961) to give a theory of pure torsion whose coupling to a spin-1/2 field formally resembled the four-fermion model of the weak interactions. In the covariant versions of the Yang–Mills-type theory the process of covariantization is vital to the existence of the symmetric part of the connection. The fact that Kibble obtained the same field equations as Weyl and Cartan is because it is not always necessary to distinguish between a coordinate transformation resulting from a change in the base and one due to a change in the affine fiber $\mathbb{R}^4$; however, in extensions of the theory and quantization (see this paper) this distinction is important; for example, in broken $\mathbb{R}^4 \circledS GL(4, \mathbb{R})$ with Poincaré gauge invariance of the second kind there would be two distinct dynamical vierbein fields were it not for the canonical form killing off the dynamical role of the field due to the postulated $\mathbb{R}^4$ invariance and reducing its significance to that of an arbitrary set of coordinate transformations.

This paper shows how reduction of the bundle of linear frames to pseudo-orthogonal frames is related to other G structures, notably where G is $GL(2,\mathbb{C})$, $CO(1,3)$, or $SL(2,\mathbb{C})$, and it shows their significance in the theory of quantization.

3. *Quantizability as a Form of Analyticity*

This section is concerned with the treatment of quantizability as a form of analyticity, and it hinges upon the relationship between light-cone and almost complex structures. Considerable use of complex structures has already been made by Segal (Segal, 1962, 1963, 1969) in an attempt to provide a mathematically consistent version of quantum field theory. The two main uses adopted by Segal were, firstly, in determining representations of $\mathbb{C}^*$ algebras on the space of holomorphic functionals on an infinite (even) dimensional pre-Hilbert space, and, more speculatively, to find a form of quantization applicable to nonlinear systems (Segal, 1963).

In the latter case Segal took the existence of a complex structure on the phase space of the system to be quantized as the fundamental feature. The special significance of the complex structure J is that on a $2n$ real dimensional manifold symplectic transformations A obey

$$A^tJA = J$$

and complex transformations satisfy

$$AJ = JA$$

so that both $GL(n,\mathbb{C})$ and $Sp(n,\mathbb{R})$ have the same maximal compact subgroup $U(n)$, and a manifold admits a symplectic structure (and hence commutator) if it admits a complex structure. This led to the idea of generalized commutation relations for nonlinear systems of the form

$$[R(z), R(z')]_{\pm} \subseteq B(z,z')1$$

where $R(z)$ is the field operator for the wave representation z and $B(z,z')$ is a bilinear form that is symmetric or antisymmetric for Fermi-Dirac and Bose-Einstein fields, respectively (and the use of the Weyl form of the commutation relations in the latter case). In Segal's work the geometry is essentially Riemannian; however, for the present application pseudo-Riemannian geometry is involved.

In comparing the light cone with these structures, the fundamental field is that of almost complex structures over space-time, M; these are in one-to-one correspondence with the elements of the homogeneous coset space $GL(4,\mathbb{R})/GL(2,\mathbb{C})$ and convert the tangent space $\mathbb{R}^4$ into a two-complex-dimensional space with respect to a given $J \in GL(4,\mathbb{R})/GL(2,\mathbb{C})$ by defining scalar multiplication by complex numbers as

$$(a + ib)X = aX + bJX, \qquad X \in \mathbb{R}^4, \quad a, b \in \mathbb{R}$$

If J is a complex structure on $\mathbb{R}^4$, then there exist elements X_1, X_2 of $\mathbb{R}^4$ such that (X_1, X_2, JX_1, JX_2) is a basis for $\mathbb{R}^4$. In particular the canonical complex structure J_0 of $\mathbb{R}^4$ induced from $\mathbb{C}^2$ maps quadruples of real numbers (x^1, x^2, y^1, y^2) into $(-y^1, -y^2, x^1, x^2)$, where $z^k = x^k + iy^k, k = 1, 2$, are coordinates of points in $\mathbb{C}^2$. In terms of the natural basis for $\mathbb{R}^4$ the canonical complex structure J_0 is represented by

$$J_0 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

where I_2 is the two-dimensional unit matrix. The real representation of the $GL(2,\mathbb{C})$ subgroup of $GL(4, \mathbb{R})$ commuting with J_0 is given by

$$(A + iB) \rightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

where $(A + iB) \in GL(2,\mathbb{C})$ and A, B are 2 x 2 real matrices.

Riemannian Hermitian metrics $h(X, Y)$ are defined as Riemannian metrics $g(X, Y)$ invariant with respect to J:

$$\{h(X, Y)\} = \{g(X, Y) | g(JX, JY) = g(X, Y)\}$$

There is then a one-to-one correspondence between the sets of Riemannian Hermitian metrics on $\mathbb{R}^4$ for a given J and the elements of the homogeneous space $GL(2,\mathbb{C})/U(2)$. The Hermitian metrics so defined are complex valued with respect to J and satisfy the three conditions below:

(1)

$$h(\lambda_1 X_1 + \lambda_2 X_2, Y) = \lambda_1 h(X_1, Y) + \lambda_2 h(X_2, Y), \qquad \lambda_1, \lambda_2 \in \mathbb{R}, \quad X_i, \quad Y \in \mathbb{R}^4$$

(2) $$h(Y, X) = \overline{h(X, Y)}$$

where the bar denotes complex conjugation

(3) $$h(JX, Y) = ih(X, Y)$$

$h(X, Y)$ can therefore be split into real and imaginary parts that are symmetric and antisymmetric, respectively:

$$h(X, Y) = S(X, Y) + iA(X, Y)$$
$$S(X, Y) = S(Y, X), A(X, Y) = -A(Y, X) \tag{3.1}$$
$$A(X, Y) = -S(JX, Y)$$

Since the space Λ^2 $(\mathbb{R}^{4*})$ of exterior 2-forms over $\mathbb{R}^4$ is isomorphic to the space of all antisymmetric bilinear forms over $\mathbb{R}^4$ there is an almost symplectic form, the almost Kahler form, corresponding to $A(X, Y)$. It is this decomposition that forms the geometric basis of Segal's quantization scheme when applied to an infinite even dimensional phase space instead of $\mathbb{R}^4$, with creation and annihilation operators $C(z)$ and $C^*(z)$ defined by

$$C(z) = (1/\sqrt{2})(z - iJz), \qquad C^*(z) = (1/\sqrt{2})(z + iJz)$$

In the case of a Riemannian structure the metric defined by the reduction $GL(4, \mathbb{R})/O(4)$ is not sufficient to determine the Hermitian structure, which as a $U(2)$ structure is not covariant with respect to $O(4)$; however, for the pseudo-Riemannian geometry of general relativity this difficulty no longer occurs.

For this the metrics are determined by the reduction $GL(4, \mathbb{R})/O(1, 3)$ and the components $g_{\mu\nu}$ defined through the pseudosymmetric vierbein $\{L^a{}_\mu\}$

$$g_{\mu\nu} = \{L^a{}_\mu\}\,\{L_{a\nu}\}$$

where by pseudosymmetry is meant the relation

$$\{L^a{}_\mu\}\eta_{a\nu} = \{L^a{}_\nu\}\eta_{a\mu}$$

and η_{ab} is the Minkowski metric $(1, -1, -1, -1)$.

For the Riemannian case the $\{L^a{}_\mu\}$ can be used as coordinates on $GL(4, \mathbb{R})/O(4)$ corresponding to the unique polar decomposition of $GL(4, \mathbb{R})$ as the product of a positive symmetric matrix with an element of the maximal compact subgroup $O(4)$ (Chevalley, 1946), but for $GL(4, \mathbb{R})/O(1, 3)$ this is only locally valid (Isham *et al.*, 1971b) and is an example of the general non-uniqueness problem, emphasized by Joseph and Solomon (Joseph and Solomon, 1970), associated with nonlinear realizations of a group G on a homogeneous coset space G/H when H is noncompact.

The reduction $GL(4, \mathbb{R})/GL(2, \mathbb{C})$ is the same as before, but the pseudo-Hermitian metrics are given by $GL(4, \mathbb{R})/\{GL(2, \mathbb{C}) \cap O(1, 3)\} \approx GL(4, \mathbb{R})/SL(2, \mathbb{C})$, i.e., by the reduction of the bundle of linear frames to that of unimodular complex linear frames. Since in general relativity $SL(2, \mathbb{C})$ appears as the structure group of the bundle of spin frames we consider the relationship between complex structures and Penrose spinors. The latter are determined by a real null direction and a phase angle, so the correspondence is defined essentially by a Wick rotation $ix^4 \leftrightarrow x^0$ converting complex unit vectors z^i $(i = 1, 2)$ into null vectors of the local flat space metric, and complex coordinates into null coordinates

$$\begin{aligned}
\sqrt{2}z^1 &= x^3 + ix^4 \leftrightarrow \sqrt{2}l^\mu = x^3 + x^0 \\
\sqrt{2}\bar{z}^1 &= x^3 - ix^4 \leftrightarrow \sqrt{2}n^\mu = x^3 - x^0 \\
\sqrt{2}z^2 &= x^1 + ix^2 \leftrightarrow \sqrt{2}m^\mu = x^1 + ix^2 \\
\sqrt{2}\bar{z}^2 &= x^1 - ix^2 \leftrightarrow \sqrt{2}\bar{m}^\mu = x^1 - ix^2
\end{aligned} \tag{3.2}$$

This differs from the Penrose formalism by the orientation of n^μ, where the $(l^\mu, n^\mu; m^\mu, m^\mu)$ constitute a null tetrad, each x^μ being a four-component vector that is related to a spinor basis (κ, ι) of $SL(2,\mathbb{C})$ by

$$l^\mu = \sigma^\mu_{B\dot{X}}\kappa^B\bar{\kappa}^{\dot{X}}$$
$$n^\mu = \sigma^\mu_{B\dot{X}}\iota^B\bar{\iota}^{\dot{X}}$$
$$m^\mu = \sigma^\mu_{B\dot{X}}\kappa^B\bar{\iota}^{\dot{X}}$$
$$m^\mu = \sigma^\mu_{B\dot{X}}\iota^B\bar{\kappa}^{\dot{X}}$$

where bar denotes complex conjugation and dot indicates a dotted spinor, whilst l^μ and n^μ are real null vectors. The present approach differs by replacing the 16 components of $\sigma^\mu_{B\dot{X}}$ by the 8 components $J^\mu_{B\dot{X}}$ determining a $GL(2,\mathbb{C})$ basis and the two components of $GL(2,\mathbb{C})/SL(2,\mathbb{C})$, so that the resultant spinors transform nonlinearly under $GL(4,\mathbb{R})$, but the equivalence of linear and nonlinear spinors has already been shown (Isham *et al.*, 1971b).

In the Penrose spinor formalism the metric components can be expressed as

$$g_{\mu\nu} = \epsilon_{AC}\epsilon_{\dot{B}\dot{D}}\sigma^{A\dot{B}}_\mu\sigma^{C\dot{D}}_\nu \qquad (3.3)$$

where ϵ_{AC} is the Levi-Civita symbol and is antisymmetric. In the nonlinear realization approach the second factorization, $GL(2,\mathbb{C}) \to SL(2,\mathbb{C})$, can be effected linearly by the two elements $\{\rho, e^{i\phi}\}$ of the multiplicative group of complex numbers $\mathbb{C}^*$ since they commute with the whole of $SL(2,\mathbb{C})$; hence the pseudosymmetric vierbein $\{L^\mu{}_a\}$ can be expressed by

$$\{L^\mu{}_a\} = \rho e^{i\phi}J^\mu_{B\dot{X}}$$

Dotted and undotted spinors transform in the opposite sense under $e^{i\phi}$, and co- and contravariant indices contragrediently under ρ, so

$$g_{\mu\nu} = \rho^{-2}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}}J^{A\dot{B}}_\mu J^{C\dot{D}}_\nu \qquad (3.4)$$

is the nonlinear spinor expression for the metric. The invariance of $g_{\mu\nu}$ under the phase transformations corresponds to the required hermiticity, and has previously been noted by Lubkin (Lubkin, 1963) who suggested a tentative identification of these operations with the $U(1)$ group of electromagnetism, whilst Weyl's unified theory employed invariance under ρ.

The two-valued nature of spinors, which is obscured in the use of complex tensors, results from the isomorphism $SL(2,\mathbb{C})/Z_2 \approx SO(1,3)$ so that the "rotation through 2π" occurs when ϕ takes the value 2π in $GL(2,\mathbb{C})$ and has real generator J in $GL(4,\mathbb{R})$. Also both J and the conjugate complex structure $-J$ determine the same reduction of $GL(4,\mathbb{R})$ to $O(1,3)$, whilst in $SL(2,\mathbb{C})$ the roles of dotted and undotted spinors are interchanged, so if J-charge is defined to be the number of undotted spinors minus that of dotted, then $J \to -J$ induces J-charge conjugation.

Closer comparison of this with the Newman-Penrose treatment of general relativity (Newman and Penrose, 1962) and with $SL(2,\mathbb{C})$ covariant flat space field theory requires extension of the usual little group classification from $\mathbb{R}^4 \circledS SL(2,\mathbb{C})$ to $GL(2,\mathbb{C})$ and $GL(4,\mathbb{R})$. The induced representation method

TABLE 1. Little groups.

Type of orbit	Little group in $GL(4,\mathbb{R})$	Little group in $GL(2,\mathbb{C})$
(1) $m^2 = 0, \quad p = 0$	$GL(4,\mathbb{R})$	$GL(2,\mathbb{C})$
(2) $m^2 > 0$	$GL(3,\mathbb{R})$	$U(2)$
(3) $m^2 = 0, \quad p \neq 0$	$GL(3,\mathbb{R})$	F
(4) $m^2 < 0$	$GL(3,\mathbb{R})$	$U(1,1)$

of Mackey involves the use of representations of $\mathbb{R}^4 \circledS K$, where K is the little group for a representative point on the orbits of the homogeneous group, G, on the character space $\hat{\mathbb{R}}^4$ of $\mathbb{R}^4$, followed by extension of the representation to the whole group by allowing the latter to act transitively on vector functions $f(g)$ on G with values in the Hilbert space of the representation of $\mathbb{R}^4 \circledS K$

$$g' : f(g) \to f(gg'), \qquad g, g' \in G$$

The points at issue here are first what happens to the $SL(2,\mathbb{C})$ orbit classification as the group is enlarged, which is given in Table I, and second the lack of manifest $SL(2,\mathbb{C})$ covariance of the representations.

If a free field $\psi(x)$ is constructed in flat space from creation and annihilation operators for zero-mass helicity-two particles

$$\psi(x) = \int \frac{d^3p}{p_0} \{e^{ipx}a(p)\psi(p) + e^{-ipx}b^+(p)\bar{\psi}(p)\}$$

where $a(p)$ and $b^+(p)$ are creation and annihilation operators and $\psi(p)$ is manifestly covariant

$$\psi(p) \xrightarrow{g} S(\Lambda)\psi(p\Lambda)e^{ip\cdot a}$$

where $g \equiv (a, \Lambda) \in \mathbb{R}^4 \circledS SL(2,\mathbb{C})$ and $S(\Lambda)$ is a finite-dimensional representation of $SL(2,\mathbb{C})$. In general $S(\Lambda)$ is arbitrary and contains subsidiary components that must be projected out by contraction with the momentum vector (Fierz, 1939), leading to the free field equations as a criterion of irreducibility; however, even then the different $S(\Lambda)$ may behave differently in presence of interactions. Instead of this, which in the case of quantum gravity leads to the reappearance of the lower spin quanta in closed loop diagrams for a Lorentz-invariant and unitary S matrix (Feynman, 1963; de Witt, 1967), with formal justification by the Faddeev–Popov method (Faddeev and Popov, 1967), it is possible to use the Weinberg approach of starting with irreducible wave functions (Weinberg, 1964a), avoiding the need for free field equations. Weinberg showed that manifestly $SL(2,\mathbb{C})$-covariant wave functions could be constructed from creation and annihilation operators for zero-mass helicity-λ particles only for $SL(2,\mathbb{C})$ representations (j_1, j_2) satisfying $j_2 - j_1 = \lambda$ as a result of the lack of semisimplicity of the relevant little group $E(2)$, and that use of the $(1, 1)$ representation for gravity led to Lorentz invariant S-matrix elements for soft-graviton scattering only if universality of the coupling was also assumed, whilst the actual wave function was not manifestly covariant. If the geometric interpretation is assumed, as here, to be more fundamental than the idea of zero-mass gravitons, then the manifest Lorentz covariance of the field is essential and consideration of the spinor field equations of general relativity

$$KT_{ABC'D'} = \Phi_{ABC'D'} + (3\Lambda - \tfrac{1}{2}C)\epsilon_{AB}\epsilon_{C'D'} \tag{3.5}$$

shows that it is the $(0,0)$ and $(1,1)$ parts of the gravitational field, Λ and $\Phi_{ABC'D'}$, that must be quantized, where K is a measure of the gravitational coupling and $T_{ABC'D'}$ the energy-momentum spinor. The Weyl spinor of type $(0,2)\oplus(2,0)$, on the other hand, does not obviously require to be quantized, although it is it that describes the free gravitational field.

The usual $SL(2,\mathbb{C})$-invariant classification of orbits does not extend to $GL(4,\mathbb{R})$; the distinction between massive, massless, and tachyonic orbits is preserved by $\mathbb{R}\times SL(2,\mathbb{C})$ and for Hermitian energy vectors by $GL(2,\mathbb{C})$, but in discussing mass it is then necessary also to consider the tensor $T_{i\bar{j}}$ given by derivation by $\partial_i\partial_{\bar{j}}$ instead of just $\Box^2$. Further extension of the group changes J, in particular if the velocity of light is regarded as a conversion factor between units of space and time it will be changed, thus giving space-time a variable refractive index, and extension also mixes all nonzero momentum of the same 4-orientation under $GL^+(4,\mathbb{R})$ and the orientations themselves in $GL(4,\mathbb{R})$. The usual distinction between positive and negative energies requires both an overall 4-orientation (sign of the dilaton) and a separate time orientability of fiberings $V^3\times\mathbb{R}^1$ of space-time defined by a globally timelike Killing field, of which only the former is necessary for the existence of an almost complex structure J. The discrete operations P and T can be described in terms of changes of these orientations, and if it were correct to identify the $U(1)$ gauge group of electromagnetism with that generated by J then CPT would be entirely geometric in origin; however, such a simple identification leads to incorrect charges.

In Table I F denotes the group of triangular matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \qquad a,b,d\in\mathbb{C}, \quad |a| = 1$$

If this classification is used as the basis for a "quantization" of gravity, then case (1) of zero-energy-momentum particles corresponds most naturally to classical geometry. There are no gauge constraints resulting from the little group classification, no distinction is made between past and future, and quantization consists of the analysis of geometric objects in terms of their holomorphic and antiholomorphic constituents with respect to a given complex structure J (if J is integrable). Propagation of gravity is described classically in this case by the hyperbolic field equations, whilst nonlinearity means that gravitons for different J are not simultaneously countable (physically they correspond to different systems of causality and measurement), and in fact arbitrary J anticommute with the canonical complex structure J_0 in the matrix representation.

The three other cases are more physical as they have nonzero energy momentum, and the first question to consider is the actual definition of this quantity in curved space-time. That adopted in the fiber bundle approach is the local flat-space definition using the affine fibers $\mathbb{R}^4$ and $\hat{\mathbb{R}}^4$, so that geometrically the energy-momentum vector contributes to a choice of coordinates and hence to $g_{\mu\nu}$ only through the constraint variables.

As in the ADM technique a (local) 3 + 1 decomposition of the gravitational field is made such that the 3-metric h_{ij} $(1 \leqslant i, j \leqslant 3)$ is given by

$$h_{ij} = \{l^a{}_i\}\ \{l_{aj}\}$$

where $\{l^a{}_i\}$ is the restriction of $\{l^a{}_\mu\}$ to the appropriate $GL(3,\mathbb{R})$ subgroup. Here the $GL(3,\mathbb{R})$ subgroup is the little group of some vector $\{N^\mu\} \in \mathbb{R}^4$ with flat-space components $N^\mu = \delta^{\mu\nu}$ for some specific ν, $0 \leqslant \nu \leqslant 3$, whilst in the ADM technique the reduction $g_{\mu\nu} \to h_{ij}$ is achieved by the choice of the four components of the constraint vector $\{(-g^{00})^{-1/2}, g_{0i}\}$ describing how to pass from one hypersurface (usually spacelike) to another by paths that are timelike, lightlike, or spacelike according to the length of the constraint vector; the gauge problem of ADM is then to find a coordinate condition reducing the six components of h_{ij} to two helicity states in a Lorentz-covariant fashion. The usual requirement that h_{ij} be transverse and tracefree is not manifestly Lorentz covariant, but corresponds in the covariant quantization scheme to inclusion in the Lagrangian of the Lagrange multiplier gauge-breaking terms $A^i h^j{}_{i,j}$ and $Bh^i{}_i$ (subject further to the Faddeev–Popov constraint for formal quantum gauge invariance) giving rise to the fictitious particles in closed loops. In the classical ADM approach the constraint vectors themselves appear in the Lagrangian as multipliers and have the physical significance of determining how a given coordinate system will be continued off a t = const hypersurface.

In the fiber bundle approach the first requirement is that the little group of N^μ in $GL(2,\mathbb{C})$ also be contained in the correct $GL(3,\mathbb{R})$ subgroup, which means that if the 3-hypersurface is to be spacelike, then N^μ must be timelike with respect to the given J. A nonlinear [with respect to $GL(4,\mathbb{R})$ transformations] graviton is then determined by a momentum vector in the direction of N^μ with energy scaled by ρ, and the single component of $U(2)/SU(2)$ left in the reduction $GL(2,\mathbb{C})/SU(2)$ using the choice of determinant and boost as above. Owing to nonlinearity this graviton is coordinate dependent; however, by the usual theory of constraints the extrinsic geometry of the hypersurface is independent of the choice of constraint vector, and hence in this model also independent of the momentum vector. Manifest general covariance requires the use of linear representations, i.e., tensors and tensor densities induced from $U(2)$ acting on the appropriate orbit in $\hat{\mathbb{R}}^4$. Instead of the six components of h_{ij} there is just a $U(2)$ tensor of type (1, 1) corresponding to the Euclidean metric which transforms under $SU(2)$ as a spin-2 object since dotted and undotted spinors are equivalent in that group. Lorentz covariantization is achieved by the Wigner boost and gives the Minkowski metric, so that the curvature comes entirely from the scalar dilaton field and the complex structure $J^{\mu}{}_{B\dot{X}}$. The dilaton can be quantized as a massive, massless, or tachyonic scalar with respect to lightcones compatible with J according to the constraint vector to be used, and physically the choice will depend on the type of matter contributing to $T_{\mu\nu}$, but in any event the noncommutativity of different J means that the $GL(4, \mathbb{R})$ covariant theory will always contain an uncountable number of off mass-shell dilatons.

The usual approach to gravity treats the free graviton as a zero-mass particle determining the conformal structure of space-time, so we consider the above model from this point of view and its relation to a Penrose spinor. As a G structure a conformal structure is a $CO(1, 3)$ structure and hence determined through the pseudosymmetric vierbein by $\{L^{\mu}{}_a\}/\text{Det}\{L^{\mu}{}_a\}$ or the pair (J, ϕ). The case of a lightlike graviton can be described as a Penrose spinor if the scale factor ρ is absorbed into the definition of the energy, the elements $\{\rho, \phi\}$ of $F/E(2) \approx \mathbb{C}^*$ being related to the phase and magnitude of the spinor by comparison of the little group classification with the null rotations for a real null direction l^{μ}. Given a pair of $SL(2, \mathbb{C})$ spinors (κ_A, ι_A) such that

$$\kappa_A \iota^A = 1 \leftrightarrow \epsilon_{AB} = \kappa_A \iota_B - \iota_A \kappa_B$$

the most general transformations of this basis which preserve the real null direction of κ^A are the null rotations (Pirani, 1964)

$$\kappa^A \to \kappa'^A = \sqrt{r} e^{i\theta/2} \kappa^A$$

$$\iota^A \to \iota'^A = (1/\sqrt{r} e^{i\theta/2})(\iota^A - rB\kappa^A)$$

where $r, \theta \in \mathbb{R}; r > 0$ and $B \in \mathbb{C}$. These elements form a four-parameter subgroup of $SL(2, \mathbb{C})$ consisting of triangular matrices

$$\begin{pmatrix} z & b \\ 0 & z^{-1} \end{pmatrix}$$

where $z = (re^{i\theta})^{1/2}$, with the restriction $|z| = 1$ giving the energy-preserving zero-mass little group $E(2)$; the phase $e^{i\theta}$ is the generator of the helicity subgroup $SO(2)$ of $E(2)$. Extending the null rotations from $SL(2, \mathbb{C})$ to $GL(2, \mathbb{C})$ introduces the two elements $\{\rho, \phi\}$ such that

$$\kappa^A \to \kappa'^A = (\rho e^{i\phi})^{1/2} \kappa^A$$

$$\iota^A \to \iota'^A = (\rho e^{i\phi})^{1/2} \iota^A$$

whilst dotted spinors transform by the complex conjugate, and lower index spinors in the opposite sense under ρ. Of the two phase transformations $e^{i\theta}$ and $e^{i\phi}$ it is the latter that is required if the spinor is to provide an invariant definition of angles and hence the conformal structure. The difference between a lightlike vierbein graviton and a Penrose spinor is thus that the former transforms according to the adjoint representation of $\mathbb{C}^* \approx GL(2, \mathbb{C})/SL(2, \mathbb{C})$ and the latter by spin 1/2, whilst restriction from $GL(2, \mathbb{C})$ to $SL(2, \mathbb{C})$ renders the phase arbitrary. Physically this graviton is a Goldstone boson complex scalar particle (ρ, ϕ) with momentum vector corresponding to a real null direction. The Segal type of quantization can be used to define formal creation and annihilation operators $C^*(z)$ and $C(z)$ for null vectors z:

$$C(z) = (1/\sqrt{2})\,(z + iJz),\; C^*(z) = (1/\sqrt{2})\,(z - iJz)$$
$$[C(z), C(z')] = 0 = [C^*(z), C^*(z')] \tag{3.6}$$
$$[C(z), C^*(z')] = A(z, z')$$

where $A(z, z')$ is the imaginary part of the Hermitian metric. The "number operator" $C(z)\,C^*(z)$ is therefore quantized in multiples of ρ^2, the square of the basic unit of length defined by the dilaton, and this will be discussed further later on in relation to points as quanta of space-time. In addition the field equations of general relativity require that ρ itself be quantized as an orthodox scalar field with causal function characterized by a specific J, whilst J remains classical, and the usual Planck's constant.

4. *Quantization of Space-Time*

In this section we consider more fully the quantization of space-time briefly mentioned at the end of the last section. The quantization of the vector space structure of the tangent space $T_x(M)$ at some point $x \in M$ in terms of creation and annihilation operators obeying (3.6) simply describes the theory of measurement when a finite unit of length is introduced in a Lorentz-covariant fashion. This quantization differs slightly in form from the ordinary commutation relations since the right-hand side is no longer a delta function, and this gives modified commutation relations between the number operator and those for creation and annihilation:

$$[N(z), C(z')] = C(z)A(z, z')$$
$$[N(z), C^*(z')] = -C^*(z)A(z, z')$$

so that except in the local flat-space metric orthogonal vectors are not simultaneously countable, and for that case the formalism resembles the indefinite Hilbert space metric quantization of a zero-mass vector field.

The preceding commutation relations are in the Fock space form, but to see the corresponding uncertainty relations we use the well-known Bargmann (Bargmann, 1961) approach to obtain the appropriate Schrödinger representation. Here the Fock function space Φ is that of entire functions on $\mathbb{C}^2$ and the Schrödinger representation space is $L^2(\mathbb{R}^2, \mu)$ where μ is the two-dimensional volume element induced by the metric. The problem considered by Bargmann was first to find a positive real function $F(x,y)$ defining an inner product in Fock space and a kernel $B_2\,(z, q)$ defining a unitary mapping of the Schrödinger representation space onto that of Fock such that

$$(f, g) = \int f(z)g(z)F(x,y)d^2z, \qquad f, g \in \Phi$$
$$f(z) = \int B_2(z, q)\psi(q)d^2q \qquad \psi(q) \in L^2(\mathbb{R}^2, d^2q)$$

The adjointness of creation and annihilation operators requires

$$(z_k f, g) = (f, \partial g/\partial z_k), \qquad 1 \leqslant k \leqslant 2$$

leading to a suitable $F(x, y)$ as

$$F = c \exp(-\bar{z} \cdot z), \qquad z_k = x_k + iy_k$$

where c is a constant. From another adjointness condition Bargmann obtained

$$B_2(z, q) = c' \exp \{-\tfrac{1}{2}(z^2 + q^2) + \sqrt{2} z \cdot q\}$$

with convenient choices of c and c' for the two-dimensional case as π^{-2} and $\pi^{-1/2}$ respectively.

In terms of the canonical complex structure J_0 (2.2) the set $\{q\}$ may be taken as (x^1, x^3) and $\{p\}$, the canonical "momenta" as (x^2, x^0) leading to the Heisenberg uncertainty relations

$$\Delta q^i \cdot \Delta p^i \sim \rho^2$$

showing that in quantum geometry measurements in directions conjugate with respect to the complex structure defined by the light cone must be liable to an error at least as great as the basic unit of length; the classical limit $\rho = 0$ is simply the case of an infinitesimal unit of length.

The quantization of curved space-time is equivalent to construction of Bargmann–Segal quantum field theory on each fiber apart from the change to an indefinite metric, with the construction varying from point to point as J and ρ vary over the base space of the fiber bundle. The vacuum is characterized by the Gaussian distribution $F(x, y)$ on $\mathbb{C}^2$ with mean square deviation determined by the dilaton field as ρ^2. The theory of principal vectors constituting an orthonormal basis for the Hilbert space of square integrable complex functions over $\mathbb{C}^2$ is just the classical theory of the Bergman kernel in this case.

Intuitively it appears plausible that a point should be the basic particle of quantum geometry and that it should have the property of either existing or not existing, and hence obey Fermi–Dirac statistics with a field of creation and annihilation operators for points defined everywhere in space-time. Such an approach is possible since according to Cartan (Cartan, 1938) a point of a three-dimensional Euclidean space can be regarded as an isotropic vector $\{x^i\}$

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = 0$$

Clearly this has only two degrees of freedom, ξ^1 and ξ^2, which may be chosen as

$$\xi^1 = \pm\sqrt{\frac{x^1 + ix^2}{2}}$$

$$\xi^2 = \pm\sqrt{\frac{x^1 - ix^2}{2}}$$

The pair (ξ^1, ξ^2) then transform under rotations according to the spin-1/2 representation of $SU(2)$. This definition depends on both a Euclidean metric and a complex structure and these must be consistent with the metric and almost complex structure for space-time. In curved space-time this construction is carried out by choosing a vector that is timelike with respect to the local light-cone structure on $T_x(M)$ and taking the 3-space as a hypersurface in $T_x(M)$ orthogonal to that vector. Quantization in terms of points is therefore equivalent to quantizing sections of the fiber bundle with structure group $SU(2)$ where the reduction $GL(2,\mathbb{C})/SU(2)$ is determined by choice of a time-like vector in $T_x(M)$, a particular phase angle ϕ and a scale factor, which is neither Lorentz covariant nor J covariant. Lorentz-covariant quantization can be carried out in the orthodox manner by postulating anticommutation relations for creation and annihilation operators for (ξ^1, ξ^2) parametrized by the direction of the normal to the hypersurface and covariant under Wigner boosts; however, for the Segal approach it is necessary to use γ matrices. Parity-conserving four-component spinors are obtained in this theory by forming the direct sum of the space $\mathbb{C}^2$ used earlier with that of a second $\mathbb{C}^2$ with opposite orientation of the 3-subspace of the corresponding real space $\mathbb{R}^4$; and if the sign of the dilaton (4-orientation) is to be preserved, then change of spatial orientation must be accompanied by a reversal of the direction of time. The γ matrices can be expressed in terms of the σ matrices defined via isotropic vectors, e.g., through the Weyl representation

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where 1 is the two-dimensional unit matrix. These γ matrices, however, transform nonlinearly under $GL(4, \mathbb{R})$ since the quantity i has the real representation J, and in this the theory resembles an attempt to remove the ambiguity of i in the Dirac equation by defining it through the pseudoscalar of the real Dirac algebra (Hestenes, 1967), The $\{\gamma_\mu\}$ form a basis for a four-real-dimensional space that can be quantized in terms of eigenstates of J by the Segal technique:

$$C(\gamma_\mu) = (1/\sqrt{2})\,(\gamma_\mu - iJ\gamma_\mu)$$
$$C^*(\gamma_\mu) = (1/\sqrt{2})\,(\gamma_\mu + iJ\gamma_\mu)$$

which has the effect of transforming the anticommutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

into the canonical light-cone form

$$\{\gamma_+, \gamma_-\} = \rho^{-2}\eta_{+-}, \qquad \{\gamma_i, \gamma_{\bar{i}}\} = \rho^{-2}\eta_{i\bar{i}}$$

where the Minkowski metric η is in off-diagonal form with only the two indicated terms (and their transposes) being nonzero. Dependence on the scale of length is eliminated to give a pure number operator if particle-antiparticle conjugation is extended to include co- and contravariance:

$$\{\gamma_\mu, \gamma^\nu\} = 2\delta^\mu_{\ \nu}$$

The approach here differs slightly from that of Hestenes, where that author attempted to interpret the phase of a real Dirac spinor as defining a mixing of particle and antiparticle states with phase angles $2n\pi$ and $(2n + 1)\pi$ denoting the respective pure states and thereby avoid the need for second quantization; that result was in conflict with the superposition principle of quantum mechanics, and here it is apparent that it is the sign of the eigenvalue of J that distinguishes between particles and antiparticles rather than the phase angle.

As with the Bose-Einstein case the vacuum can be characterized by a distribution of variance ρ^2, but for the spinors the real Hilbert space is the $\mathbb{R}^4$ of the γ matrices embodying both parities and the distribution D skew-symmetric, i.e., a Clifford distribution (Segal, 1956)

$$D(\xi)D(\xi^1) + D(\xi^1)D(\xi) = 2\rho^2\eta(\xi, \xi^1), \qquad \xi, \xi^1 \in \mathbb{R}^4$$

Canonical position and momentum operators act on the space $L^2(\mathbb{R}^4, D)$ by left multiplication by $D(\xi)/\sqrt{2}\rho$ and by $iD(\xi)/\sqrt{2}\rho$ followed by the automorphism of $L^2(\mathbb{R}^4, D)$ induced by reflection in a hyperplane orthogonal to ξ, respectively.

Instead of describing points as isotropic vectors it is also possible to define them geometrically in terms of pairs of twistors (Penrose, 1968), so a twistor analog of the present argument is given using second-order G structures (Kobayashi, 1972). Selecting a flat hypersurface with isotropic vectors defined by points a Möbius construction can be carried out so that $O(1, 4)$ is the resulting Möbius group, which is covered by the symplectic group $Sp(1, 1)$ whose generators can be constructed out of pairs of creation and annihilation operators. The 3-D analog of twistors are basis vectors for the lowest spin representation of the latter and can be subduced from the twistor group $SU(2, 2)$ which covers the Möbius group $O(2, 4)$ of the four-dimensional space. When a base space is defined prerelativistically there is little advantage in using twistors, and their geometric significance is that, subject to a $\mathbb{Z}_4$ cohomological consistency condition, they define fields of conformal spinor connection coefficients corresponding to the reduction $G^2(4)/O(2,4)$ of the second-order frame bundle, where

$$G^2(4) \equiv \{a_j^{\ i}, a^i_{jk}; a_j^{\ i} \in GL(4, \mathbb{R}), a^i_{jk} = a^i_{kj}\}$$

5. *Application of Kostant's Technique to General Relativity*

In this section we consider the application of Kostant's technique of quantization (Kostant, 1970a, b) to general relativity. This approach is intimately related to that of the previous section, but whereas the Bargmann-Segal quantization only required almost complex and almost Hermitian structures leading to a generally covariant form of quantum mechanics, for Kostant's method integrability is required to give a Kähler manifold structure; physically this is clearly necessary for a field quantization to be invariant under parallel transport.

Essentially Kostant extends the process of quantizing functions representing classical mechanical properties on a homogeneous symplectic G space and uses it to classify representations of the group G. If F is a space of functions over a manifold M with symplectic structure Ω where the functions form a Lie algebra under the Poisson bracket operation, then quantization results from using the 2-form Ω to define a map from F into the space of vector fields on M. If $f, g \in F$ are function on M with P, B $\{f, g\}$ then the corresponding quantum operators are X_f and X_g, where

$$-\iota_{X_f}\Omega = df \tag{5.1}$$

ι_X denotes interior multiplication by X, and

$$\{f, g\} = \iota_{X_g} \cdot \iota_{X_f}\Omega = 2\Omega(X_f, X_g) \tag{5.2}$$

The mapping $f \to X_f$ defines a Lie algebra homomorphism whose kernel consists of the constant functions, and for the example to be treated here it is related to the vacuum state. In addition Kostant imposes the Bohr-Sommerfeld type of integrality condition that the 2-cocycles on M with respect to Ω must have integer values, corresponding to the condition that Ω be the curvature form of some line bundle, which in general requires certain cohomology conditions to be met (Kostant, 1970b). For the case of general relativity a principal line bundle with structure group $GL(2, \mathbb{C})/SL(2, \mathbb{C})$ arises naturally leading to integrality when the symplectic form is taken to be the curvature form of this bundle, i.e., the Ricci form, whilst the homogeneity condition must be dropped if the results are not to be trivial physically.

The requirement that the almost complex structure J be integrable means that its Nijenhuis torsion $N(X, Y)$ must be zero, which is automatically satisfied if the ordinary torsion $T(X, Y)$ of the almost complex connection is zero by virtue of the relation (Kobayashi and Nomizu, 1970)

$$T(JX, JY) - J(T(JX, Y)) - J(T(X, Y)) - T(X, Y) = -\tfrac{1}{2}N(X, Y)$$

so for general relativity, but not the Einstein-Cartan theory, there is no loss of generality in assuming integrability.

If a scale of length is to be everywhere defined, the function determining the gravitational field when restricted to the line bundle must be nonvanishing, and if complex analyticity is fundamental to quantization, sections of the bundle must be holomorphic. If A denotes the sheaf of germs of holomorphic functions, A^* that of nonvanishing holomorphic functions, and $\mathbb{Z}$ that of integers, then there is an exact sequence of sheaves

$$0 \to \mathbb{Z} \to A \xrightarrow{\exp} A^* \to 1$$

(and similarly for real 1-forms) where exp: $f \mapsto \exp(2\pi i f), f \in A$, and there is a corresponding exact sequence of cohomology groups

$$\to H^1(M, \mathbb{Z}) \to H^1(M, A) \to H^1(M, A^*) \to H^2(M, \mathbb{Z}) \to \cdots$$

where M denotes the base space of the bundle. The cohomological equivalence classes of gravitational fields having a given J (conformal structure modulo a phase factor) correspond by definition to the elements of $H^1(M, A^*)$. In the work of Hestenes (Hestenes, 1967) an attempt was made to define a scale of length through spinors describing matter, but this led to difficulties through the implicit use of elements of $H^1(M, A)$ instead of $H^1(M, A^*)$. Since A is known to be a fine sheaf, i.e.,

$$H^q(M, A) = 0, \qquad q \geqslant 1$$

the equivalence classes are uniquely determined by their images in $H^2(M, \mathbb{Z})$, the first Chern class. In the particular case of a Kähler structure the Ricci 2-form is exact, so that the line bundle is a product bundle and use of cohomology is unnecessary; however, this would not be so for the more general situation considered in the previous section. The Ricci form, S, of a Kähler manifold is given in the local geodesic coordinates which naturally exist in this case by

$$S = \sum_{i,j} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

where

$$R_{i\bar{j}} = (1/2\pi i)\partial_i \partial_{\bar{j}} \rho^2$$

and ρ, which is real, is just the scale of length.

Kostant uses the Ricci form where nonvanishing, as the symplectic form Ω in (5.1) and considers only the quantization of real functions, which for classical mechanics where M is phase space involves no loss of generality since observables are assumed to be real, whereas for the present application physical reality corresponds to hermiticity. Kostant obtains a commutative diagram of Lie algebra homomorphisms

$$\begin{array}{ccccc} & & e(L, \nabla) & & \\ & \nearrow & & \searrow & \\ 0 \to \mathbb{R} & & & & H(M) \to 0 \\ & \searrow & & \nearrow & \\ & & C^\infty_{\mathbb{R}}(M) & & \end{array}$$

where $\mathbb{R}$ denotes the real constant functions on M, $C^\infty_{\mathbb{R}}(M)$ the real smooth functions on M, $H(M)$ the vector fields on M that are globally Hamiltonian with respect to Ω, and

$$e(L, \nabla) = \left\{ \eta \left| \begin{array}{l} \eta \text{ is a real vector field on the line bundle } L \text{ such} \\ \text{that the connection 1-form } \omega \text{ and } \rho^2 \text{ are invariant} \\ \text{along the flow of } \eta, \text{ and } \eta \text{ is invariant under } \mathbb{C}^* \end{array} \right. \right\}$$

where ∇ denotes covariant differentiation corresponding to ω, and L has structure group $GL(2,\mathbb{C})/SL(2,\mathbb{C}) \approx \mathbb{C}^*$, where $\mathbb{C}^* \equiv \mathbb{C} - \{0\}$. The locally Hamiltonian vector fields $e(L, \nabla)$ are equivalent to $C^\infty_{\mathbb{R}}(M)$ as a central extension of $H(M)$: for $\psi \in C_{\mathbb{R}}{}^\infty(M)$ and $\eta_\psi \in e(L, \nabla)$

$$\psi(x) = -\langle \omega, \eta_\psi \rangle_v, \qquad x \in M, \quad v \in L_{\pi^{-1}(x)}$$

which by $\mathbb{C}^*$ invariance of η is independent of v. The map (5.1) $\psi \mapsto \xi_\psi$; $\psi \in C_{\mathbb{R}}{}^\infty(M)$, $\xi_\psi \in H(M)$ defines an action of ψ on the space $\Gamma(L)$ of sections of L by covariant derivation. Because of curvature this does not lead directly to a structure of $\Gamma(L)$ as an $H(M)$ module, so Kostant defines prequantization as

$$\delta : \mathbb{C}_{\mathbb{R}}{}^\infty(M) \to \text{End}\,\Gamma(L)$$

$$\delta : \psi s \mapsto (\nabla_{\xi_\psi} + 2\pi i \psi)s, \qquad s \in \Gamma(L), \quad \psi \in \mathbb{C}_{\mathbb{R}}{}^\infty(M) \tag{5.3}$$

which does preserve the symplectic form, as also does the classical action of the functions

$$(\psi s)(x) = \psi(x)s(x)$$

For Kostant's quantization it is necessary to make $\Gamma(L)$ into a Hilbert space and to separate the action of derivation from multiplication by ψ in (5.3), which is achieved by the process of polarization. In the trivial case of $M \equiv \mathbb{R}^4 \approx \mathbb{C}^2$ with the natural Kähler structure prequantization is unnecessary and the theory of $\Gamma(L)$ as a Hilbert space is just the well-known Bergman kernel theory.

A polarization (Kostant, 1970b) is a C^∞ choice of a subspace F_x of the complexified tangent space $T_x{}^{\mathbb{C}}(M)$, $\forall\, x \in M$ such that the following conditions hold:

(1) F_x is a vector subspace of $T_x{}^{\mathbb{C}}(M)$ that is maximally isotropic with respect to Ω_x, i.e.,

$$\Omega_x(F_x, F_x) = 0$$

(2)

$$U_F(M) = \{\xi \in U(M) \mid \xi_x \in F_x,\ \forall\, x \in M\}$$

is a Lie subalgebra of the algebra $U(M)$ of complex vector fields on M.

(3) Dim $(F_x \cap \bar{F}_x) = k$, where k is a constant for all x, and the bar denotes complex conjugation. The actual choice of F_x is such as to give a set of directions $\{\xi_x\} \in F_x$ in which the covariant derivative vanishes, but usually the choice is not unique, leading to a major equivalence problem solved to date only for a few special cases, e.g., different real polarizations (Blattner *et al.*, 1974).

Direct application of Kostant technique to space-time corresponds to taking $T_x(M)$ as $\mathbb{R}^4$, complexifying it, and looking for a line bundle to use in quantization, but in the present approach the complex structure is defined nonlinearly via the reduction $GL(4,\mathbb{R})/GL(2,\mathbb{C})$, and the assumption that quantization is a form of analyticity corresponds to a particular choice of polarization of the real tangent space $T_x(M)$, namely, the Kähler polarization, which satisfies $F_x \cap \bar{F}_x = 0$. Using the local geodesic coordinates vanishing of the covariant derivative in this case is just the vanishing of the derivative of a holomorphic function with respect to an antiholomorphic vector field. The polarization problem in this case amounts to showing the equivalence of the Fock space and $L^2(\mathbb{R}^2, dx^2)$ representations mentioned earlier, which is achieved for local flat space by the Bargmann-Segal work (Bargmann, 1961).

To make the global sections $\Gamma(L)$ into a Hilbert space it is necessary to introduce the complex volume element leading to a sign ambiguity in taking its square root. In the Kostant theory a bundle B_F of bases of F is defined as a principal fiber bundle over F with structure group $GL(n, \mathbb{C})$ where n = Dim F, which for the present application is the bundle used from the outset with $n = 2$, and a double $\tilde{B}^F$, the metalinear frame bundle, of B^F corresponding to the sign ambiguity in taking the square root of the determinant in $GL(n, \mathbb{C})$. A space of half-forms $L^F{}_x$

$$L^F{}_x \equiv \{v : \tilde{B}^F{}_x \to \mathbb{C} \mid v(bg) = \chi(g)^{-1} v(b)\}, \quad \forall\, b \in \tilde{B}^F{}_x$$

is defined and used to form a line bundle L^F of half-forms

$$L^F = \bigcup_{x \in M} L^F{}_x$$

The wave functions W^F of the quantum theory are taken to be square integrable functions ψ, where

$$\psi \equiv \{\psi \in \Gamma(L \otimes L^F) \mid \nabla_\xi \psi = 0, \forall \xi \in U_F(M)\}$$

For the existence of the double covering $\tilde{B}^F$ it is necessary that $H^2(M, \mathbb{Z}_2)$ vanish (Simms, 1974). For general relativity the sign ambiguity is that which appears in the transformation of spin-1/2 objects under dilatations (for a fixed sign of the dilaton) and which appears in flat space theory in the definition of the energy as the square root of the rest mass. Geometrically it appears in the global existence condition for a spinor structure (Lichnerowicz, 1968) resulting from the exact sequence of sheaves of local sections corresponding to

$$0 \to \mathbb{Z}_2 \to SL(\underset{\sim}{2, \mathbb{C}}) \overset{p}{\to} O(\underset{\sim}{1, 3}) \to 0$$

$$\to H^1(M, SL(\underset{\sim}{2, \mathbb{C}})) \overset{p^*}{\to} H^1(M, O(\underset{\sim}{1, 3})) \overset{\delta^*}{\to} H^2(M, \mathbb{Z}_2) \to$$

where δ^* is induced by the coboundary operator. If $E \in H^1(M, O(\underset{\sim}{1, 3}))$ corresponds to a spin structure S

$$E = p^* S$$

then $\delta^* E = 0$ since the sequence is exact, thereby requiring the vanishing of the second Whitney class. In a physical interpretation this is necessary for a global distinction between particles and antiparticles.

The curvature tensor of L for general relativity is the Ricci tensor $R_{i\bar{j}}$ and can be expressed directly in terms of the Penrose curvature spinors Λ and

$\Phi_{ABC'D'}$ for the same complex structure

$$R_{i\bar{j}} \leftrightarrow 6\Lambda\epsilon_{AB}\epsilon_{C'D'} - 2\Phi_{ABC'D'}$$

whilst $L \otimes L^F$ involves the dual of the curvature. The Hamiltonian vector fields are those that preserve $R_{i\bar{j}}$ and J, the latter being an extremely restrictive criterion, and they will only exist for problems having some symmetry. In the trivial case of flat space all vector fields are Hamiltonian for the Kähler form, and Kostant's theory can be applied to give the usual representation theory of $\mathbb{R}^4 ⓢ SL(2,\mathbb{C})$, whilst more generally Killing vector fields can be considered but no global quantization is possible for a realistic matter distribution, and in that case the nearest approach is the use of asymptotic Killing vectors and representations of the BMS group. Alternatively the quantum mechanical approach of the previous section can be applied at each point of space-time and analyticity considered for the sections of L. Since momentum states are no longer considered the lack of commutation of dilatations with momenta is no longer a drawback to using representations of $GL(2,\mathbb{C})$. Homogeneous functions of degree (n_1, n_2) are analytic if the difference $(n_1 - n_2)$ is an integer corresponding to J-charge quantization, whilst if the scale function is also to be analytic n_1 and hence n_2 must also be integral restricting representations to polynomials, i.e., the spinor representations. The contribution of massive spinors to the breaking of dilatation invariance can be carried out locally in the analytic case by expanding ρ as a function of the two complex variables (z_1, z_2) and (assuming the validity of ordinary quantum mechanics) taking the expansion coefficients as functions of the Compton wavelength of particles belonging to the appropriate representation present at that spot. Since $R_{i\bar{j}}$ defines the Hamiltonian form, other quantum numbers that can be assigned are those denoting eigenstates of gauge symmetries, and in particular of the generalized duality rotations of $\Phi_{ABC'D'}$ (Plebanski, 1964).

For the case in which Hamiltonian vector fields do exist the Kostant technique can be used to "quantize" the Ricci form $R_{i\bar{j}}dz^i \wedge dz^j$ as a "quantized" differential form in the sense of Segal (Segal, 1968). The basic definition introduced by Segal is of a quantized differential form of degree k over a real linear vector space V equipped with a nondegenerate antisymmetric bilinear form Ω as a k-linear mapping from V into the Weyl algebra $\mathbf{E}$ over the pair (V, Ω), and a O-form is a member of $\mathbf{E}$. For the present application the definition must either be extended to complex vector spaces, or else the Schrödinger representation used for V as $\mathbb{R}^4$ with the real polarization. In the latter case $R_{\mu\nu}$ is a 2-form and Ω is the Kähler form on $\mathbb{R}^4$ corresponding to the local light-cone, and the mapping (5.1) $\psi \mapsto \xi_\psi$ maps the real values of $R_{\mu\nu}$ into elements of the Weyl algebra using the given polarization. The Weyl algebra for this flat space is that obtained by Bargmann (Bargmann, 1961) in considering the inhomogeneous unitary transformations of $\mathbb{C}^n$ with $n = 2$ and amounts to writing the commutation relations in the Weyl form, whilst the Weyl algebra for the curved space amounts to the same thing for the Hamiltonian vector fields with respect to the Ricci form.

6. *Discussion*

The essential difference between the schemes of quantization considered here and in usual flat space theories is that the latter type of theory is now defined vertically on the affine fibers isomorphous to $\mathbb{R}^4$ but not horizontally. Instead of defining creation and annihilation operators for global fields in terms of their Fourier components local creation and annihilation operators have been defined in terms of the almost complex structure at each point, so that in general the definition of a given type of particle varies throughout space-time. The Fock space at each point consists of entire functions of two complex variables, so that states with a high occupation number are characterized as single-particle states of a high spin field. In this paper the precise relationship between the nonlinear spinor formalism and the description of field quanta in terms of momentum states has not been considered as it will be covered in a separate paper on the non-Lagrangian coupling of matter to geometry.

The whole approach of this paper depends on the existence of almost complex structures on space-time, and whilst the necessity of even dimensionality and orientability have been pointed out, they are not sufficient existence conditions. For compact manifolds study of the homomorphism on cohomology rings induced by the mapping $h: G(m, N, \mathbb{C}) \to G(2m, N, \mathbb{R})$ of Grassmann spaces defined by taking to an m-complex-dimensional vector space of $E_{m+N}(\mathbb{C})$ its underlying real vector space leads to the following existence conditions (Wu, 1952):

$$W^i = 0 \qquad (i \text{ odd})$$

$$\sum_{0 \leqslant i \leqslant [m/2]} (-1)^i p_i = \sum_{0 \leqslant i \leqslant m} (-1)^i c_i \sum_{0 \leqslant j \leqslant m} c_j$$

where W^i and p_i denote the Stiefel-Whitney and Pontryagin characteristic classes of the real vector bundle and c_k the Chern classes of the corresponding unitary $U(m)$ buncle. For space-time applications this leads to the incompatibility of the quantization scheme with an underlying S^4 topology.

For the case of an integrable almost complex structure and quantization of sections of the line bundle an extreme limitation arises for compact manifolds from the fact that if a function is holomorphic it is necessarily a constant, rendering the whole quantization scheme trivial.

In the covariant quantum mechanical situation of Section 4 there is no globally consistent quantization and both Nijenhuis and the ordinary torsion tensor will in general be nonzero, so that at a strictly classical level there is a problem over causality since for nonzero torsion causal structures deduced from congruences of null and timelike geodesics no longer coincide. The Ricci tensor for the line bundle L no longer has the simple Kähler form but still corresponds up to a scale factor to the same symplectic structure as the light-cone. So its use as a Hamiltonian for general relativistic quantum mechanics seems plausible, in which case its eigenvectors as in classical gravitational radiation would feature as quantum states as well as bases for representations of the generalized duality rotations; this together with the nonlinear spinors will be considered elsewhere in relation to unified field theories in the old Einstein sense.

Finally since quantization has been presented here as a form of analyticity it seems reasonable if the continuum is fundamental to postulate the existence of fields that are not quantizable in this way. Two aspects of this arise, one the straightforward extension of the domain of allowable representations of $SL(2,\mathbb{C})$ from polynomials to homogeneous functions lacking J-charge quantization and having in general infinite dimension, and the other the possibility of relating divergences of the energy-momentum tensor to poles of the complex scale factor.

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